Note on the Development of an Algebraic Fraction.

By Capt. P. A. MacMahon, R. A.

In the *American Journal of Mathematics*, Vol. V, No. 3, M. Faà de Bruno has considered the development, in ascending powers of x, of the algebraic fraction

$$\phi(x) = \frac{1}{1 + a_1 x + a_2 x^2 + \ldots + a_n x^n},$$

and has obtained the coefficient of x^p in the form of a determinant.

His result may be simply obtained as follows.

For convenience I take the fraction to be

$$f(x) = \frac{1}{1 - a_1 x + a_2 x^2 - \ldots + (-)^n a_n x^n}.$$

Let

$$\begin{aligned} F(y) &= y^n - a_1 y^{n-1} + a_2 y^{n-2} - \ldots + (-)^n a_n \\ &= (y-\alpha)(y-\beta)(y-\gamma)\ldots \end{aligned}$$

so that $\alpha, \beta, \gamma \ldots$ are the roots of the equation $F(y) = 0$;

then

$$\frac{1}{y-\alpha} = \frac{1}{y} + \frac{\alpha}{y^2} + \frac{\alpha^2}{y^3} + \ldots$$

and

$$\frac{1}{F(y)} = \frac{1}{y^n} + \frac{H_1}{y^{n+1}} + \frac{H_2}{y^{n+2}} + \ldots + \frac{H_p}{y^{n+p}} + \ldots$$

wherein H_p represents the sum of the homogeneous symmetric functions, of weight p, of the roots of the equation

$$F(y) = 0.$$

Write now $y = \frac{1}{x}$ and divide both sides of the resulting equation by x^n, thus obtaining

$$\frac{1}{1 - a_1 x + a_2 x^2 - \ldots + (-)^n a_n x^n} = 1 + H_1 x + H_2 x^2 + \ldots + H_p x^p + \ldots$$

It is well known that

$$H_p = \begin{vmatrix} a_1 & a_2 & a_3 & \ldots & & a_p \\ 1 & a_1 & a_2 & \ldots & & a_{p-1} \\ 0 & 1 & a_1 & \ldots & & a_{p-2} \\ 0 & 0 & 1 & \ldots & & a_{p-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & a_1 \end{vmatrix}$$

which is equivalent to M. Faà de Bruno's result.

Now $$H_p = \Sigma (-)^{p+\alpha_1+\alpha_2+\alpha_3+\dots} \frac{(\alpha_1+\alpha_2+\alpha_3+\dots)!}{\alpha_1!\,\alpha_2!\,\alpha_3!\dots} a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3}\dots$$
the summation extending to all integer, including zero, solutions of the equation $$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots = p;$$
consequently we have the result
$$\frac{1}{1 - a_1 x + a_2 x^2 - \dots + (-)^n a_n x^n}$$
$$= \sum_{p=\infty}^{p=0} \sum (-)^{p+k} \frac{k!}{\alpha_1!\,\alpha_2!\,\alpha_3!\dots} a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} \dots x^p,$$
where $$\alpha_1 + \alpha_2 + \alpha_3 + \dots = k,$$
$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots = p.$$

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NOTE BY DR. FRANKLIN.

The general coefficient in the expansion of
$$\frac{1}{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n}$$
is obviously given immediately in the form of a determinant by comparison of coefficients. If the required series is
$$b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n,$$
we have
$$(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n)(b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n) = 1,$$
whence
$$\begin{aligned} a_0 b_0 &= 1 \\ a_1 b_0 + a_0 b_1 &= 0 \\ a_2 b_0 + a_1 b_1 + a_0 b_2 &= 0, \text{ etc.,} \end{aligned}$$
whence, solving for the b's,
$$b_p = (-)^p \left(\frac{1}{a_0}\right)^{p+1} \begin{vmatrix} a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_p & a_{p-1} & \cdots & \cdots & a_0 \end{vmatrix}.$$
The above is so obvious that I have been in the habit of regarding it as the natural method of obtaining the value of H_p, whereas, in the preceding note, Captain MacMahon has reversed the process.